ON A CHANG CONJECTURE

BY

RALF-DIETER SCHINDLER

Mathematisches Institut, Universität Bonn
Beringstr. 4, D-53115 Bonn, Germany
e-mail: rds@rhein.iam.uni-bonn.de

ABSTRACT

We use the core model for one strong cardinal to show that the Chang Conjecture $(\aleph_{n+2}, \aleph_{n+1}) \Rightarrow (\aleph_{n+1}, \aleph_n)$ together with $2^{\aleph_{n-1}} = \aleph_n$ implies, for $1 < n < \omega$, the existence of an inner model with a strong cardinal. An essential step of our proof is an application of the Gitik Game which also admits a presentation.

Let us first recall the definition.

Definition 1: Let κ be an infinite cardinal. $(\kappa^{++}, \kappa^+) \Rightarrow (\kappa^+, \kappa)$ denotes the following model theoretic assertion. For any structure $\mathfrak{A} = (A; P, \ldots)$ of finite type s.t. A is the universe of $\mathfrak{A}$, $P \subset A$ is a distinguished predicate, and $\mathrm{Card}(A) = \kappa^{++}$, and $\mathrm{Card}(P) = \kappa^+$, there is $\mathfrak{B} = (B; P \cap B, \ldots)$ s.t. $\mathfrak{B} \prec_{\Sigma_\omega} \mathfrak{A}$, $\mathrm{Card}(B) = \kappa^+$, and $\mathrm{Card}(P \cap B) = \kappa$.

Silver has shown that the Erdös cardinal $\kappa(\omega_1)$ can be used to get the Chang Conjecture $(\aleph_2, \aleph_1) \Rightarrow (\aleph_1, \aleph_0)$ in a forcing extension (cf. [S]). Afterwards, Donder proved that ω_2^V is ω_1-Erdös in the core model of Dodd and Jensen provided $(\aleph_2, \aleph_1) \Rightarrow (\aleph_1, \aleph_0)$ (cf. [D]).

For $\kappa > \omega$, $(\kappa^{++}, \kappa^+) \Rightarrow (\kappa^+, \kappa)$ is much stronger. Variating a construction of Kunen, Laver showed how to get, for any $n < \omega$, $(\aleph_{n+2}, \aleph_{n+1}) \Rightarrow (\aleph_{n+1}, \aleph_n)$ by forcing using a huge cardinal (cf. [F]). On the other hand, $(\aleph_3, \aleph_2) \Rightarrow (\aleph_2, \aleph_1)$ implies the existence of 0^{sword}, i.e. a mouse M s.t. $o^M(\kappa) > 1$, some $\kappa < \mathrm{On} \cap M$ (cf. [V]). In fact, a refinement of the method of [V] might yield a mouse M s.t. $o^M(\kappa) \geq \kappa$, some $\kappa < \mathrm{On} \cap M$, but not more. A slight strengthening of

Received June 14, 1995 and in revised form September 12, 1995

$(\aleph_3, \aleph_2) \Rightarrow (\aleph_2, \aleph_1)$ can be shown to imply $o(\kappa) = \kappa^{+\omega}$, some κ, in an inner model (cf. [Sch2]). The exact consistency strength of $(\kappa^{++}, \kappa^+) \Rightarrow (\kappa^+, \kappa)$ is not yet known for any $\kappa > \omega$.

Before now stating our theorem let us remark that Foreman (in his [F], §8) sketches how to obtain, given a huge cardinal, for any $n < \omega$, an inner model of the forcing extension in which both $(\aleph_{n+2}, \aleph_{n+1}) \Rightarrow (\aleph_{n+1}, \aleph_n)$ and the G.C.H. hold.

THEOREM: *If $(\aleph_4, \aleph_3) \Rightarrow (\aleph_3, \aleph_2)$ holds and $2^{\aleph_1} = \aleph_2$ then there is an inner model with a strong cardinal.*

Our proof will work with the countably complete core model for one strong cardinal, which we denote by K^C (cf. [J], [K], [Sch1]). We shall use several properties of K^C in a "black box fashion", so that a reader not familiar with core model theory can perhaps follow our treatment. However, the following lines sketch the proofs of two essential properties of K^C that will be used. The missing details may be found in [Sch1], or may easily be derived from results in [J].

K^C is a constructible inner model, in fact a "class-sized mouse", of the form $L[E]$ where $E = (E_\alpha\colon \alpha < \mathrm{On})$ is well-organized sequence of (partial or total) extenders. By $K^C \downarrow \alpha$, $\alpha < \mathrm{On}$, we denote the initial segment $J_\alpha[E \restriction \alpha]$ of K^C. Every $K^C \downarrow \alpha$ is a mouse, and K^C has the following properties, for all $\alpha < \mathrm{On}$:

(a) If $K^C \downarrow \alpha$ has a largest cardinal κ not overlapped in K^C, and there is a mouse N end-extending $K^C \downarrow \alpha$ s.t. $\rho_N^\omega \leq \kappa$, then the minimal such N is an initial segment of K^C.

(b) If F is a countably complete extender on $K^C \downarrow \alpha$ s.t. $(K^C \downarrow \alpha, F)$ is a premouse, then $\emptyset \neq E_\alpha = F$; and, on the other hand, every $E_\alpha \neq \emptyset$ which is an extender on K^C (i.e. E_α is total) is countably complete.

(a) ensures that K^C is "saturated with so-called collapsing mice", and (b) states that all and only countably complete total extenders are included in K^C's extender sequence. Beneath standard concepts and facts from the core model theory below one strong cardinal, we shall use the following two lemmata. We denote by $\neg L^{\mathrm{strong}}$ the assumption that there is no inner model with a strong cardinal.

LEMMA 1: *Assume $\neg L^{\mathrm{strong}}$. Let κ be a regular cardinal, and let H_κ denote the set of all sets hereditarily smaller than κ. Suppose that $\pi\colon \bar{H} \to_{\Sigma_\omega} H_\kappa$ is s.t.*

$\bar{H}$ is transitive, ${}^{\omega}\bar{H} \subset \bar{H}$, and $\pi \neq \mathrm{id}$. Let $\bar{K}$ denote $(K^C)^{\bar{H}}$, i.e. the countably complete core model defined in $\bar{H}$. Then there is a mouse N end-extending $\bar{K}$ s.t. $\rho_N^\omega < \mathrm{On} \cap \bar{H}$.

Proof (Sketch): Note that $\pi \restriction \bar{K}: \bar{K} \to_{\Sigma_\omega} K^C$. Moreover, the assumption ${}^{\omega}\bar{H} \subset \bar{H}$ can be used to verify that, if N is a mouse end-extending an initial segment M of $\bar{K}$ and the upward mapping technique (cf. [J] or [Sch1]) is used to find a "nice" mapping $\tilde{\pi}: N \to Q$ s.t. $\tilde{\pi} \supset \pi \restriction M$, then Q is a mouse, in turn.

Now, using methods of [J] (cf. also [Sch1]) one verifies that $\bar{K}$ is *not* saturated with collapsing mice. Taking a minimal collapsing mouse N omitted in $\bar{K}$ one finds that the coiteration of N with $\bar{K}$ does not move $\bar{K}$. Hence some iterate of N is as desired. ∎ (Lemma 1)

LEMMA 2: *Assume $\neg L^{\text{strong}}$. Let $\kappa \in \mathrm{Card}^{K^C}$ not be overlapped in K^C, and set $\nu := O^{K^C}(\kappa) := \kappa^{+K^C}$ plus the Mitchell order of κ in K^C. Let $X \subset \nu$ be s.t. $\mathrm{Card}(X)^{\aleph_0} < \mathrm{Card}(\kappa)$. Then there is $Y \in K^C$ s.t. $Y \supset X$ and $\mathrm{Card}^{K^C}(Y) \leq \kappa$. In particular, this conclusion holds provided $2^{\aleph_0} \leq \aleph_2, \mathrm{Card}(X) = \aleph_1$, and $\mathrm{Card}(\kappa) \geq \aleph_3$.*

Proof (Sketch): The proof is by induction on $\sup(X)$. If some X is given, one may thus assume that $\alpha = \sup(X)$ is a K^C-cardinal $> \kappa$ and choose $\pi: \bar{H} \to_{\Sigma_\omega} H_{\nu^+}$ s.t. $\{\kappa\} \cup X \subset \mathrm{ran}(\pi)$, ${}^{\omega}\bar{H} \subset \bar{H}$, and π has a critical point less than κ. As in the preceding lemma, there is an end-extension N of $(K^C)^{\bar{H}} \downarrow \pi^{-1}(\alpha)$ s.t. $\rho_N^\omega < \pi^{-1}(\alpha)$. Via the upward mapping technique, let $\tilde{\pi}: N \to Q$ be s.t. $\tilde{\pi} \supset \pi \restriction (K^C)^{\bar{H}} \downarrow \pi^{-1}(\alpha)$, and Q is a mouse.

An argument is needed to show that $Q \in K^C$ (cf. [Sch1]). But then, as is standard in covering arguments, the Skolem functions of Q can be used to get, combined with the inductive hypothesis, a covering as desired. ∎ (Lemma 2)

We now commence with the proof of the theorem.

Proof of the Theorem: Let us suppose $(\aleph_4, \aleph_3) \Rightarrow (\aleph_3, \aleph_2), 2^{\aleph_1} = \aleph_2$, and $\neg L^{\text{strong}}$ to hold. We aim to derive a contradiction.

To commence, choose $(H; \in) \prec_{\Sigma_\omega} (H_{\omega_4}; \in)$ s.t. $\mathrm{Card}(H) = \aleph_4$, $\omega_4 \subset H$, and ${}^{\omega_1}H \subset H$. By the Hausdorff Formula, $\aleph_4^{\aleph_1} = \aleph_4 \cdot 2^{\aleph_1} = \aleph_4$, so this choice is possible. As well, because $\aleph_2^{\aleph_1} = \aleph_2 \cdot 2^{\aleph_1} = \aleph_2$, we may then choose $S: \omega_2 \leftrightarrow {}^{\omega_1}\omega_2$ bijective. Pick $F: \omega_4 \leftrightarrow H$ bijective. Let $G = (g_\xi: \xi < \omega_4)$ be a family of pairwise almost disjoint functions $g: \omega_3 \to \omega_3$; i.e., for all $\xi < \xi' < \omega_4$, $g_\xi(\alpha) \neq g_{\xi'}(\alpha)$ for all but $\aleph_2$ many $\alpha < \omega_3$.

Moreover, let $h\colon \omega \times [H]^{<\omega} \to H$ be a Σ_ω-Skolem function for the structure $\bar{\mathfrak{A}} := (H; \omega_3, \in, S, F, G)$, and let $P\colon [H]^{<\omega} \to \omega_3$ be defined by $P(\{x_0, \ldots, x_n\}) := \sup(h''(\omega \times [\omega_2 \cup \{x_0, \ldots, x_n\}]^{<\omega}) \cap \omega_3)$.

Now set $\mathfrak{A} := (H;\ \omega_3,\ \in,\ S,\ F,\ G,\ h,\ P)$, and use $(\aleph_4, \aleph_3) \Rightarrow (\aleph_3, \aleph_2)$ to find $\mathfrak{B} = (B;\ \omega_3 \cap B,\ \in,\ S \restriction B,\ F \restriction B,\ G \restriction B,\ h \restriction B,\ P \restriction B) \prec_{\Sigma_\omega} \mathfrak{A}$ s.t. $\mathrm{Card}(B) = \aleph_3$ and $\mathrm{Card}(\omega_3 \cap B) = \aleph_2$. Set $\delta := \sup(\omega_3 \cap B) < \omega_3$.

Let $C := h''(\omega \times [\omega_2 \cup B]^{<\omega})$. Clearly,

$$\mathfrak{C} := (C; \omega_3 \cap C, \in, S \restriction C, F \restriction C, G \restriction C) \prec_{\Sigma_\omega} \bar{\mathfrak{A}},$$

Card (C) $= \aleph_3$, and $\omega_2 \subset C$. Hence by a standard argument, $\omega_3 \cap C$ is an ordinal.

We in fact claim that $\omega_3 \cap C = \delta$. It suffices to show $\delta = \sup(\omega_3 \cap B) \geq \omega_3 \cap C$. Suppose that $\xi \in \omega_3 \cap C$. Then $\xi \in h''(\omega \times [\omega_2 \cup B]^{<\omega}) \cap \omega_3$, i.e. there are some $x_0, \ldots, x_n \in B$ s.t. $\omega_3 > P(\{x_0, \ldots, x_n\}) \geq \xi$. But of course $P(\{x_0, \ldots, x_n\}) \in B$ so that $\delta \geq \omega_3 \cap C$.

Now consider $\sigma\colon \bar{\mathfrak{B}} := (\bar{H}, \delta, \in, \bar{S}, \bar{F},\ \bar{G}) \cong \mathfrak{C} \prec_{\Sigma_\omega} \bar{\mathfrak{A}}$. We have seen that $\delta \in (\omega_2, \omega_3)$ is the critical point of σ, and clearly $\sigma(\delta) = \omega_3$. $\mathrm{On} \cap \bar{H} \geq \omega_3$, as otherwise $\mathrm{Card}(\bar{H}) = \mathrm{Card}(\bar{F}''\,\mathrm{On} \cap \bar{H}) < \aleph_3$; contradiction! Moreover, if $\mathrm{On} \cap \bar{H} > \omega_3$ then $\bar{H}$ would have at least five distinct infinite cardinals, viz., $\omega, \omega_1, \omega_2, \delta$, and ω_3; contradiction! Thus $\mathrm{On} \cap \bar{H} = \omega_3$, and $\omega, \omega_1, \omega_2$ and δ are the infinite cardinals of $\bar{H}$.

We now claim that $\mathrm{cf}^V(\delta) = \omega_2$. Suppose otherwise, i.e. that $\mathrm{cf}(\delta) < \omega_2$. Let $D \in [\delta]^{\leq \omega_1}$ be a set of ordinals $< \delta$ cofinal in δ. Let $\bar{G} = (\bar{g}_\xi\colon \xi < \omega_3)$. $\bar{\mathfrak{B}} \models$ "$\bar{G}$ is a family of pairwise disjoint functions $\bar{g}\colon \delta \to \delta$". Hence $g_\xi \restriction D \neq g_{\xi'} \restriction D$ for $\xi \neq \xi'$, $\xi, \xi' < \omega_3$, so that $\{\bar{g}_\xi \restriction D\colon \xi < \omega_3\}$ has cardinality $\aleph_3$. But $\mathrm{Card}(D) \leq \aleph_1$, and $\mathrm{ran}(\bar{g}_\xi) \subset \delta$ where $\mathrm{Card}(\delta) = \aleph_2$, so $\{\bar{g}_\xi \restriction D\colon \xi < \omega_3\}$ has at most $\aleph_2^{\aleph_1} = \aleph_2 \cdot 2^{\aleph_1} = \aleph_2$ (by $2^{\aleph_1} = \aleph_2$) many members. Contradiction! Hence really $\mathrm{cf}(\delta) = \omega_2$.

Let $X \in [\omega_2]^{\leq \omega_1}$, and choose $f\colon \omega_1 \to X$ surjective. $f \in H$ by ${}^{\omega_1}H \subset H$, and so there is $\xi < \omega_2$ with $S(\xi) = f$. But then $\bar{S}(\xi) = f$, too, because $\delta = \mathrm{c.p.}(\sigma) > \omega_2$, and so $X = f''\omega_1 \in \bar{H}$. That is, $[\omega_2]^{\leq \omega_1} \subset \bar{H}$.

We now claim that ${}^{\omega_1}\bar{H} \subset \bar{H}$. Using $\bar{F}$, it suffices to show that $[\omega_3]^{\leq \omega_1} \subset \bar{H}$. Let $X \in [\omega_3]^{\leq \omega_1}$ be arbitrary. Let $c\colon \delta \leftrightarrow \sup(X) \cup \delta$ be bijective s.t. $c \in \bar{H}$. As $\mathrm{cf}(\delta) = \omega_2$, $\xi := \sup(c^{-1}{}''X) < \delta$, and we may thus choose $d\colon \omega_2 \leftrightarrow \xi$ bijective s.t. $d \in \bar{H}$. Now $Y := d^{-1} \circ c^{-1}{}''X \in [\omega_2]^{\leq \omega_1} \subset \bar{H}$, and so $X = c \circ d''Y \in \bar{H}$.

We thus have the following situation. There is $\sigma: \bar{H} \to_{\Sigma_\omega} H_{\omega_4}$ with $\delta = \text{c.p.}(\sigma) < \omega_3, {}^{\omega_1}\bar{H} \subset \bar{H}$, and $\text{On} \cap \bar{H} = \omega_3$.

Now let $K := K^C \downarrow \omega_4$ where K^C is the countably complete core model. $\bar{K} := \sigma^{-1}{}''K$ is the countably complete core model inside $\bar{H}$, and $\sigma \restriction \bar{K}: \bar{K} \to_{\Sigma_\omega} K$. By Lemma 1, there is a mouse M end-extending $\bar{K}$ s.t. $\rho^\omega_M < \omega_3$. Let M be a minimal such in the sense of the pre-wellordering of all mice.

Let $\mathfrak{I}= (M_i\colon i \le \Theta), (\pi_{ij}\colon i \le j \le \Theta)$ be the canonical fine iteration of $M_0 = \text{core } (M)$ s.t. $M_\Theta = M$. This iteration is simple by our choice of M. As $\text{Card}(M_0) \le \aleph_2, \Theta \ge \omega_3$. We may thus assume w.l.o.g. that $\Theta = \omega_3$, as surely $M_{\omega_3} \supset \bar{K}$. Let $(\nu_i)_{i<\omega_3}$ be the index of $\mathfrak{I}$.

CLAIM 1: *There is a stationary $C \subset \omega_3$ s.t. for all $i \in C, \text{cf}(i) = \omega_1$ and $\{j < i\colon \pi_{ji}(\nu_j) = \nu_i\}$ is unbounded in i.*

Proof: Let $\Pi(i) := \{j < i\colon \pi_{ji}(\nu_j) = \nu_i\}$ for $i < \omega_3$, and suppose Claim 2 to fail. Then $C' := \{i < \omega_3\colon \text{cf}(i) = \omega_1 \text{ and } \Pi(i) \text{ is bounded in } i\}$ is stationary in ω_3. Let D be the set of all limit ordinals $i < \omega_3$ closed under Gödel's pairing function Γ and s.t. $j < i$ implies $\text{On} \cap M_j < i$. Clearly, D is club in ω_3. Hence $C'' := C' \cap D$ is stationary in ω_3.

It can then easily be checked that $f\colon C'' \to \omega_3$ is regressive, where $f(i) = \Gamma(j, \nu, \sigma)$ iff (a) j is minimal s.t. $\exists \bar{\nu} \in M_j, \pi_{ji}(\bar{\nu}) = \nu_i$, (b) $\pi_{ji}(\nu) = \nu_i$ and (c) σ is the lowest upper bound of $\Pi(i)$.

By Fodor's Theorem, let $C_0 \subset C''$ be stationary s.t. $f''C_0 = \{\Gamma(j_0, \tilde{\nu}_0, \sigma_0)\}$, and pick $i, j \in C_0$, $j < i$. Then, on the one hand, $\pi_{ki}(\nu_k) = \nu_i$ implies $k < \sigma_0 < j$. But on the other hand, $\pi_{ji}(\nu_j) = \pi_{ji}(\pi_{j_0 j}(\tilde{\nu}_0)) = \pi_{j_0 i}(\tilde{\nu}_0) = \nu_i$. Contradiction! ∎ (Claim 1)

Using Claim 1, let $(i_\xi)_{\xi \le \omega_1}$ be s.t. $i_\xi < \omega_3$ and $\pi_{i_\xi i_{\xi'}}(\nu_{i_\xi}) = \nu_{i_{\xi'}}$ for all ξ, ξ' s.t. $\xi \le \xi' \le \omega_1$. Set $F := E^{M_i}_{\nu_i}$ where $i := i_{\omega_1}$. F is an extender which is used in $\mathfrak{I}$ just after it has been used ω_1 many times before. Such an extender is "reconstructible" inside $\bar{H}$:

CLAIM 2: $F \in \bar{H}$.

Let us finish the proof of the theorem assuming Claim 2 to hold, i.e. let us postpone the verification of Claim 2 until later.

For notational convenience, set $W_\xi := M_{i_\xi}$ and $\pi'_{\xi\xi'} := \pi_{i_\xi i_{\xi'}}$ for $\xi \le \xi' \le \omega_1$, and set $\tilde{\nu}_\xi := \nu_{i_\xi}$ for $\xi \le \omega_1$. As we shall never again explicitly need other mice and mappings from $\mathfrak{I}$ we shall furthermore easily write W_i for W_ξ if $\xi = i \leq \omega_1$, π_{ji} for $\pi'_{\xi\xi'}$ if $\xi = j \leq \xi' = i \leq \omega_1$, and ν_i for $\tilde{\nu}_\xi$ if $i = \xi \leq \omega_1$. Let κ_i be the critical point of π_{ii+1}, $i < \omega_1$, and let κ be the critical point of F.

CLAIM 3: *F is countably complete.*

Proof: Let $(X_n, a_n)_{n<\omega}$ be s.t. $X_n \in F_{a_n}$ where $a_n \in [\nu_{\omega_1}]^{<\omega}$, all $n < \omega$. Choose any $i < \omega_1$ s.t. there are $\bar{X}_n, \bar{a}_n \in W_i$ with $\pi_{i\omega_1}(\bar{X}_n) = X_n$ and $\pi_{i\omega_1}(\bar{a}_n) = a_n$, all $n < \omega$. Let $\tau := \pi_{i\omega_1}{}^{-1} \restriction \bigcup_{n<\omega} a_n \colon \bigcup_{n<\omega} a_n \to \bigcup_{n<\omega} \bar{a}_n$. Clearly, τ is order preserving and $\mathrm{ran}(\tau) \subset \kappa$ by $\mathfrak{I}$'s normality. Using $\pi_{i\omega_1}, \bar{X}_n \in E^{W_i}_{\nu_i, \bar{a}_n}$, and so $\tau'' a_n = \bar{a}_n \in \pi_{i\omega_1}(\bar{X}_n) = X_n$, all $n < \omega$, again using $\mathfrak{I}$'s normality. ∎ (Claim 3)

Now it easily follows from Claim 2 and Claim 3 that $\bar{H} \models$ "$(\bar{K} \downarrow \nu_{\omega_1}, F)$ is a premouse s.t.F is countably complete". Hence by the definition of $\bar{K}$ as $(K^C)^{\bar{H}}$, $F = E^{\bar{K}}_{\nu_{\omega_1}}$. But because F was used in $\mathfrak{I}$ yielding an end-extension M of $\bar{K}$, $E^{\bar{K}}_{\nu_{\omega_1}} = E^{M}_{\nu_{\omega_1}} = \emptyset$. Contradiction!

We are left with the task of giving the

Proof of Claim 2: It is enough to show that, in $\bar{H}$, the system $(W_i \downarrow \nu_i \colon i \leq \omega_1)$, $(\pi_{ij} \restriction W_i \downarrow \nu_i \colon i \leq j \leq \omega_1)$ can be "approximated" to a sufficiently close degree. (Note that $W_i \downarrow \nu_i = \bar{K} \downarrow \nu_i$, all $i \leq \omega_1$.) I.e., it is enough to define, in $\bar{H}$, a sequence $(b_\beta)_{\beta<\nu_{\omega_1}}$ of **threads for $\beta < \nu_{\omega_1}$ in W_{ω_1}**; i.e. $b_\beta = (\gamma_{\beta,i})_{i<\omega_1}$ is a sequence of ordinals $\gamma_{\beta,i} < \kappa$ s.t. for all but countably many $i < \omega_1$, $\pi_{i\omega_1}(\gamma_{\beta,i}) = \beta$. Because assume such $(b_\beta)_{\beta<\nu_{\omega_1}}$ to be given inside $\bar{H}$. Let $a = \{\beta_1, \ldots, \beta_n\} \in [\nu_{\omega_1}]^n$, some $n < \omega$. Then, as usual, for any $X \in \wp([\kappa]^n) \cap W_{\omega_1}$, $X \in F_a$ iff for all but countably many $i < \omega_1$, $\{\gamma_{\beta_1,i}, \ldots, \gamma_{\beta_n,i}\} \in X$; i.e., F is definable in $\bar{H}$, and thus $F \in \bar{H}$ as desired.

Following Gitik (cf. his [G] p. 231ff.) we shall use the "Gitik Game" to find the correct sequence $b_\beta = (\gamma_{\beta,i})_{i<\omega_1}$ inside $\bar{H}$ (cf. also [G-M] p. 14ff.). Our presentation of this game differs slightly from those in both [G] and [G-M].

Definition 2: The **Gitik Game $\mathfrak{G}$** is defined as follows. The game is played by two players, Adam and Eve, who alternatingly make $\leq \omega$ moves. Adam starts with his 0th move. On his nth move, $n < \omega$, Adam plays a pair (B_n, h_n) where $B_n \subset \nu_{\omega_1}, \mathrm{Card}(B_n) \leq \aleph_1, B_n \supset B_{n-1}$, provided $n > 0$, and $h_n \in {}^{\kappa}\nu_{\omega_1} \cap \bar{K}$. Eve responds, on her nth move, with a sequence $(\tau_{n,i} \colon i < \omega_1)$ where

(i) $\tau_{n,i}$: $\bar{K} \downarrow \nu_i \supset\!\!\rightarrow_{\Sigma_\omega} \bar{K} \downarrow \nu_{\omega_1}$, i.e. $\tau_{n,i}$ maps a submodel of $\bar{K} \downarrow \nu_i$ elementarily into $\bar{K} \downarrow \nu_{\omega_1}$, and $\mathrm{Card}(\tau_{n,i}) \leq \aleph_1$, all $i < \omega_1$,

(ii) $\tau_{n,i} \supset \tau_{n-1,i}$ for all $i < \omega_1$, provided $n > 0$,

(iii) $B_n \cap h_n'' \kappa_i \subset \mathrm{ran}(\tau_{n,i})$ for all but countably many $i < \omega_1$.

If Eve was able to play an nth move, any $n < \omega$, then Adam replies with his $(n+1)$st move. Eve wins a play of $\mathfrak{G}$ iff the number of moves is infinite.

In a play of $\mathfrak{G}$, Eve has to present her version of a sufficiently large part of $(\bar{K} \downarrow \nu_i : i \leq \omega_1)$, $(\pi_{ij} \upharpoonright \bar{K} \downarrow \nu_i : i \leq \omega_1)$, and Adam tries to show her version not to be consistent. Note that every play of $\mathfrak{G}$ with moves from $\bar{H}$ happens to be an element of $\bar{H}$ by virtue of ${}^{\omega_1}\bar{H} \subset \bar{H}$. As well, $\mathfrak{G}$ is open and so determined, i.e. either Adam or Eve has a winning strategy (in V, as well as in $\bar{H}$).

Subclaim 1: Eve has a winning strategy for $\mathfrak{G}$ in $\bar{H}$.

Proof: Suppose not. Then Adam has a winning strategy σ in $\bar{H}$. Let us consider a play of $\mathfrak{G}$ where Adam uses σ for his choice of (B_n, h_n) on his nth move, $n < \omega$, and Eve replies with $(\tau_{n,i} : i < \omega_1)$ defined by $\tau_{n,i} := \pi_{i\omega_1} \upharpoonright H_i$, where H_i is s.t. $H_i \prec \bar{K} \downarrow \nu_i$, $\mathrm{Card}(H_i) \leq \aleph_1$, and $\pi_{i\omega_1}^{-1}{}'' B_n \subset H_i$, all $i < \omega_1$, on her nth move (chosen in V). We claim that Eve never breaks any of the rules (i)–(iii) of $\mathfrak{G}$.

Let $n < \omega$ be arbitrary. (i) and (ii) are trivial. Let us check (iii). As the tail of the iteration $\mathfrak{I}$ from W_{ω_1} to the end-extension of $\bar{K}$ is beyond ν_{ω_1}, $h_n \in {}^{\kappa}\nu_{\omega_1} \cap \bar{K}$ implies $h_n \in W_{\omega_1}$ (cf. [K] 4.2 (iii)). We have to verify $B_n \cap h_n'' \kappa_i \subset \mathrm{ran}(\pi_{i\omega_1})$ for all but countably many $i < \omega_1$. Let $i < \omega_1$ be s.t. there is $\bar{h} \in W_i$, $h = \pi_{i\omega_1}(\bar{h})$. Let $\xi \in B_n \cap h_n'' \kappa_i$ be arbitrary, and let $\bar{\xi} < \kappa_i$ be s.t. $\xi = h(\bar{\xi})$. Then $\xi = h(\bar{\xi}) = \pi_{i\omega_1}(\bar{h})(\bar{\xi}) = \pi_{i\omega_1}(\bar{h}(\bar{\xi})) \in \mathrm{ran}(\pi_{i\omega_1})$.

Hence this play of $\mathfrak{G}$ is infinite, contradicting the fact that Adam used the winning strategy σ for choosing his moves. ∎ (Subclaim 1)

Let σ be any of Eve's winning strategies, and let $\beta < \nu_{\omega_1}$. Consider a play of $\mathfrak{G}$ in which $B_0 = \{\beta\}$, and $\mathrm{ran}(h_0) = \{\beta\}$, and Eve plays according to σ. Suppose $(\tau_{0,i} : i < \omega_1)$ to be her first move. Then $\beta \in \mathrm{ran}(\tau_{0,i})$ for all but countably many $i < \omega_1$. We then let $\sigma^\frown\beta$ denote some $(\gamma_i)_{i<\omega_1}$ s.t. $\gamma_i = \tau_{0,i}^{-1}(\beta)$ for any $i < \omega_1$ with $\beta \in \mathrm{ran}(\tau_{0,i})$.

Let $c = (\gamma_i)_{i<\omega_1}$, $d = (\delta_i)_{i<\omega_1}$ be sequences of ordinals. We write $c \leq d (c < d$, resp.) iff for all but countably many $i < \omega_1$, $\gamma_i \leq \delta_i$ ($\gamma_i < \delta_i$, resp.). (I.e., $<$ denotes "eventual dominance".)

Subclaim 2: If $b = (\beta_i\colon i < \omega_1) \not\geq$ the thread for β in W_{ω_1}, some $\beta < \nu_{\omega_1}$, then $b \neq \sigma^\frown\beta$ for any of Eve's winning strategies σ.

Proof: Suppose not, and let σ be one of Eve's winning strategies with $\sigma^\frown\beta = b$, some $\beta < \nu_{\omega_1}$ and $b = (\beta_i\colon i < \omega_1) \not\geq$ the thread for β in W_{ω_1}. Let us consider a play of $\mathfrak{G}$ where $B_0 = \{\beta\} = \text{ran}(h_0)$, Eve uses σ for her choice of $(\tau_{n,i}\colon i < \omega_1)$ on her nth move, $n < \omega$, and Adam plays as follows. Let $n > 0$. First he sets $B_n := \bigcup_{i<\omega_1} \pi''_{i\omega_1} \text{dom}(\tau_{n-1,i}) \in [\nu_{\omega_1}]^{\leq\omega_1} \subset \bar{H}$ (chosen in V). Then, by Lemma 2 applied inside $\bar{H}$, there is $h \in \bar{K}$ s.t. $h''\kappa \supset B_n$, and Adam sets $h_n := h$ for some such h. We claim that

(#) Adam wins this play.

Proof: Suppose not. Then the play is infinite.

Again by [K] 4.2 (iii), $h \in W_{\omega_1}$. Let $i < \omega_1$ be s.t. there is $\bar{h} \in W_i$ with $\pi_{i\omega_1}(\bar{h}) = h$. If $\xi \in B_n \cap \text{ran}(\pi_{i\omega_1})$, $\xi = \pi_{i\omega_1}(\bar{\xi})$, say, then:

$$W_{\omega_1} \models \exists\tilde{\xi} < \kappa\ h(\tilde{\xi}) = \xi,$$

so

$$W_i \models \exists\tilde{\xi} < \kappa_i\ \bar{h}(\tilde{\xi}) = \bar{\xi},$$

i.e., for some $\tilde{\xi} < \kappa_i$, $W_i \models \bar{h}(\tilde{\xi}) = \bar{\xi}$, so

$$W_{\omega_1} \models h(\tilde{\xi}) = \xi.$$

Hence $B_n \cap \text{ran}(\pi_{i\omega_1}) \subset h''\kappa_i$ for such i. So by rule (iii) for Eve, $B_n \cap \text{ran}(\pi_{i\omega_1}) \subset \text{ran}(\tau_{n,i})$ for all but countably many $i < \omega_1$, all $n < \omega$. Pick $i^* < \omega_1$ s.t. for all $n < \omega$, $B_n \cap \text{ran}(\pi_{i^*\omega_1}) \subset \text{ran}(\tau_{n,i^*})$ and s.t. $\pi_{i^*\omega_1}(\beta_{i^*}) < \beta$ and $\tau_{0,i^*}(\beta_{i^*}) = \beta$.

Now let us inductively define $(\eta_n\colon n < \omega)$ by setting $\eta_0 := \pi_{i^*\omega_1}(\beta_{i^*}) < \beta$ and $\eta_n := \pi_{i^*\omega_1} \circ \tau^{-1}_{n-1,i^*}(\eta_{n-1})$ for $n > 0$. Note that if $0 < n < \omega$ and η_{n-1} is well-defined, then $\eta_{n-1} \in B_{n-1} \cap \text{ran}(\pi_{i^*\omega_1})$ and η_n is well-defined also by $B_{n-1} \cap \text{ran}(\pi_{i^*\omega_1}) \subset \text{ran}(\tau_{n-1,i^*})$. We claim that $\eta_n < \eta_{n-1}$ for all $n < \omega$, $n > 0$.

As $\tau_{0,i^*}(\beta_{i^*}) = \beta$, $\eta_1 = \pi_{i^*\omega_1} \circ \tau^{-1}_{0,i^*}(\eta_0) < \pi_{i^*\omega_1}(\beta_{i^*}) = \eta_0$. If $\eta_n < \eta_{n-1}$, some $n < \omega$, $n > 0$, then $\tau^{-1}_{n,i^*}(\eta_n) < \tau^{-1}_{n,i^*}(\eta_{n-1}) = \tau^{-1}_{n-1,i^*}(\eta_{n-1})$ by (i) and (ii) of Eve's rules, and thus $\eta_{n+1} = \pi_{i^*\omega_1} \circ \tau^{-1}_{n,i^*}(\eta_n) < \pi_{i^*\omega_1} \circ \tau^{-1}_{n-1,i^*}(\eta_{n-1}) = \eta_n$.

We have found an $\in$-descending sequence of ordinals of length ω. Contradiction! ∎ (#)

But now we have reached a contradiction, because Adam wins this play although Eve follows her winning strategy σ. ∎ (Subclaim 2)

It can easily be checked now that for any given $\beta < \nu_{w_1}$, Eve has, in $\bar{H}$, a winning strategy σ s.t. $\sigma^\frown\beta$ is a thread for β in W_{w_1}. Hence using Subclaims 1 and 2, the thread b_β may be defined inside $\bar{H}$ as a $\leq$-minimum of $\{\sigma^\frown\beta: \sigma$ is one of Eve's winning strategies$\}$. ∎ (Claim 2)

∎ (Theorem)

Of course, there is a wide gap between strong and huge cardinals, and the consistency strength of $(\aleph_4, \aleph_3) \Rightarrow (\aleph_3, \aleph_2) + 2^{\aleph_1} = \aleph_2$ lies somewhere between them. Moreover, the method of our proof yields the little bit generalized result from the abstract. But it does not yield L^{strong} from $(\aleph_3, \aleph_2) \Rightarrow (\aleph_2, \aleph_1)+$ C.H. which would require one to reconstruct, in $\bar{H}$, an extender which has been used only ω many times. [Sch2] yields more than $o(\kappa) \geq \kappa^{+\omega}$ in an inner model if the latter Chang Conjecture + C.H. holds.

ACKNOWLEDGEMENT: I am indebted to Professor Peter Koepke for innumerable discussions, explanations, and hints on this and related topics. Professor Menachem Magidor let me know the trick, due to Professor Foreman and himself, how to obtain an ω_1-closed submodel as above. I gratefully acknowledge a fellowship of the Graduiertenkolleg Mathematik at the University of Bonn. The current paper is a revised chapter from my Thesis [Sch1] on the core model with Professor Koepke.

References

[D] H.-D. Donder, R.B. Jensen and B.J. Koppelberg, *Some applications of the core model*, in *Set Theory and Model Theory* (Bonn, 1979), Lecture Notes in Mathematics **872**, Springer, Berlin, 1981.

[F] M. Foreman, *Large cardinals and strong model theoretic transfer properties*, Transactions of the American Mathematical Society **272** (1982), 427–463.

[G] M. Gitik, *On measurable cardinals violating the continuum hypothesis*, Annals of Pure and Applied Logic **63** (1993), 227–240.

[G-M] M. Gitik and W. Mitchell, *Indiscernible sequences for extenders, and the singular cardinal hypothesis*, to appear.

[J] R. Jensen, *The core model for non-overlapping extender sequences*, handwritten notes, Oxford.

[K] P. Koepke, *Fine structure for inner models with strong cardinals*, Habilitation, Freiburg i.B., 1989.

[Sch1] R.-D. Schindler, *The core model up to one strong cardinal*, Ph.D. Thesis, Bonn, 1996.

[Sch2] R.-D. Schindler, *On a Chang Conjecture, II*, submitted.

[S] J. Silver, *The independence of Kurepa's conjecture and two-cardinal conjectures in model theory*, in *Axiomatic Set Theory*, Proceedings of Symposia in Pure Mathematics **13**, I (D. Scott, ed.), American Mathematical Society, Providence, R.I., 1971, pp. 383–390.

[V] J. Vickers, D.Phil. Thesis, Oxford, 1994.